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MEASURABLE GROUP ACTIONS ARE ESSENTIALLY BOREL ACTIONS

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ABSTRACT

If a locally compact group G acts on a Lebesgue probability space (X, λ), it is natural to consider these conditions: (a) each group element preserves the class of λ, and (b) the action function is measurable. The latter is a weakening of the requirement that the action be Borel, provided X has a particular Borel structure as well as the σ-algebra of measurable sets. In this paper, we give an example showing that such an action need not be Borel relative to the given Borel structure, and prove that there is always a conull invariant subset and a new standard Borel structure on that subset for which the action is Borel. This is the meaning of the title.

0. Introduction

When studying actions of locally compact groups, which we always assume to be second countable, topological actions and Borel actions occur and are natural types to consider. When a quasiinvariant measure is present, it also seems natural to consider broadening the context to include measurability, giving the measure or its class more prominence. Here the first notion involved is to allow mappings under which inverse images of Borel sets are not required to be Borel sets, but only measurable, i.e., in the completion of the Borel sets relative to the measure in question. Once the measure is completed, we notice that it is possible for it to be the completion of its restriction to more than one σ-algebra of "Borel sets". Thus we can start with a measure and a Borel space and complete it, or start with a complete measure and find compatible Borel structures. To put more weight on the measure in our definition of measurable actions, we take the latter point of departure. The conclusion of the theorem will be that with the proper choice of Borel structure we can obtain a standard Borel G-space on an

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invariant set that is conull in the given space. In particular, from the measure theoretic viewpoint, no generality is lost by studying only Borel actions.

All details, including definitions, are given later, but here we give an outline of the argument: Let $(X, \mathcal{M}, \lambda)$ be a Lebesgue probability space and let G be a locally compact group. If $a: X \times G \to X$ is an action of G on X, on the right, we may write xg for $a(x, g)$. For a to be measurable, we require first that every $a(\cdot, g)$ be a measurable automorphism of $(X, \mathcal{M}, [\lambda])$, i.e. preserving the measure class $[\lambda]$. Then G acts on the measure algebra $M(\lambda)$, by automorphisms, and on $L^2(\lambda)$, by unitary operators. For the latter we use a Radon–Nikodym derivative to compensate for the fact that a given $g \in G$ may not preserve λ. If we identify $M(\lambda)$ with a set of projections on $L^2(\lambda)$, the action of G on $M(\lambda)$ is implemented by the unitary operators. In fact we have a system of imprimitivity. If a is measurable, that unitary representation is measurable and hence continuous. If the action of G on $M(\lambda)$ is even weakly measurable, then there is a compact topological G-space Y and a Borel measure μ on Y so that $M(\mu)$ and $M(\lambda)$ are isomorphic as G-spaces. Taking first any map $\psi_1: Y \to X$ which implements this isomorphism, we find ψ_1 can be changed on a null set to get an equivariant measurable map ψ. Then ψ carries a conull invariant Borel set in Y one-one onto a conull invariant set in X, and that gives the desired Borel structure. This is what it means for the given action to be essentially Borel. By proper choice of invariant subset and Borel structure, whenever the techniques for Borel space actions would be useful, they are available.

It may be worth noticing that the theorem we use says we can even have this subset isomorphic to a Borel subset in a compact metric space on which the action is continuous [7, Theorem 7.5.5], but we do not know that it is isomorphic to a closed set, or even a G_δ. It would be interesting to know if the Borel set can be given a Polish topology, maintaining continuity of the action. The issue here is that the group may not be discrete, since for discrete groups such a result is known [12].

After giving two formulations of the definition of measurable action, we give an example showing that there exist Borel structures compatible with a given measure class for which the action a is not Borel. Of course, the positive result of the paper is that good choices of Borel structure can also be made.

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1. Definitions, an Example and the Theorem

It is common in ergodic theory to study transformations of Lebesgue spaces, and we will follow that tradition. One way to say that $(X, \mathcal{M}, \lambda)$ is a *Lebesgue* (probability) *space* is to say that it is isomorphic modulo null sets to the unit interval with Lebesgue measurable sets and Lebesgue measure [2, Appendix 1; 10; 11, page 2]. From this it follows that there is a countably generated σ-algebra, $\mathcal{B}_1$, contained in $\mathcal{M}$ such that

(1) λ is the completion of $\lambda \mid \mathcal{B}_1$ and there is a conull set $X_1 \subseteq X$ such that $(X_1, \mathcal{B}_1 \mid X_1)$ is a standard Borel space.

Thus, for any such $\mathcal{B}_1$, $(X, \mathcal{B}_1, \lambda \mid \mathcal{B}_1)$ is metrically standard [4]. Also, every $\mathcal{M}$-measurable function on X agrees a.e. with a $\mathcal{B}_1$-measurable function, and for any property of points in X that holds a.e. there is a conull element of $\mathcal{B}_1$ on which it holds. This set may even be taken to be a subset of X_1, and hence standard as a Borel space.

We take the strict definition of automorphism for a Lebesgue space $(X, \mathcal{M}, [\lambda])$ or $(X, [\lambda])$: a *measurable automorphism* is a function f taking X one-one onto X such that $f^{-1}(E) \in \mathcal{M}$ when $E \in \mathcal{M}$, and the image measure $f(\lambda)$ is equivalent to λ. Thus f preserves $\mathcal{M}$ and $[\lambda]$, or f is non-singular. We could allow f to be a bijection between conull subsets, but for a general group of transformations it is best to use the strict definition. If $\mathcal{B}_1$ is a σ-algebra satisfying (1), and $f\colon X \to X$ is one-one and onto and such that $f^{-1}(E) \in \mathcal{M}$ whenever $E \in \mathcal{B}_1$, we can define $f(\lambda) \mid \mathcal{B}_1$. If $f(\lambda) \mid \mathcal{B}_1 \sim \lambda \mid \mathcal{B}_1$, it follows easily that $f^{-1}(E) \in \mathcal{M}$ whenever $E \in \mathcal{M}$. If Y is a Borel space and $f\colon X \to Y$, we call f *λ-measurable* provided $f^{-1}(B) \in \mathcal{M}$ whenever $B \subseteq Y$ is Borel. This terminology is chosen to emphasize the distinction between λ-measurable functions and Borel (measurable) functions.

An automorphism of $(X, [\lambda])$ induces a measure algebra automorphism. The measure algebra $M(\lambda)$ is defined to be $\mathcal{M}/\lambda^{-1}(0)$, measurable sets modulo null sets. Since $\mathcal{M}$ is obtained from $\mathcal{B}_1$ by adjoining null sets, $M(\lambda)$ is also $\mathcal{B}_1$/null sets. Now let q be the quotient homomorphism of $\mathcal{M}$ onto $M(\lambda)$. Then $f^*(q(E)) = q(f^{-1}(E))$ for $E \in \mathcal{M}$ defines an automorphism of $M(\lambda)$.

Let G be a locally compact group. Then there is a finite Borel measure ν on G such that ν is equivalent to Haar measure and symmetric: $\nu(B^{-1}) = \nu(B)$ for every Borel set B contained in G. Let $a\colon X \times G \to X$ be a (right) action of G on X. We want to discuss possible definitions of the measurability of this action. The first condition we impose is

(M1) for g in G, $a(\cdot, g)$ is a measurable automorphism of $(X, [\lambda])$.

Since a is our action, each $a(\cdot, g)$ is one-one and onto, so all (M1) adds is measurability of the function and its behavior relative to $[\lambda]$.

Condition (M1) says that the action gives a homomorphism (or anti-homomorphism) of G into the group $\mathrm{Aut}(X, [\lambda])$, but it does not require any continuity or measurability of that homomorphism. There are at least two ways to add such a condition. The first we mention is a natural generalization of the usual definition of a Borel action:

(M2) a is $\lambda \times \nu$-measurable.

Note that this depends only on the classes of λ and ν. By itself, (M2) implies that almost every $a(\cdot, g)$ is measurable, but says nothing about how these maps move λ. Together (M1) and (M2) seem to make reasonable conditions for defining measurability of a, or the notion of a measurable group of automorphisms of $(X, [\lambda])$.

Definition. *If* $(X, \mathcal{M}, \lambda)$ is a Lebesgue space, G is a locally compact group and $a : X \times G \to X$ is an action of G on X, we say a is measurable iff (M1) and (M2) are satisfied.

For later use, we point out a single condition which is equivalent to the combination of (M1) and (M2). The statement involves an involution τ of $X \times G$ defined by $\tau(x, g) = (a(x, g), g^{-1})$. This is the inversion map of the groupoid $X \times G$.

Lemma 1. *The action a is measurable iff* (M) *τ is a measurable automorphism of* $(X \times G, [\lambda \times \nu])$.

Proof. Since the coordinate projections are Borel functions, τ is $\lambda \times \nu$-measurable iff a is. Write λg for the image measure $a(\cdot, g)(\lambda)$. Then

$$\tau(\lambda \times \nu) = \int (\lambda g \times \varepsilon_{g^{-1}}) d\nu(g) = \int (\lambda g^{-1} \times \varepsilon_g) d\nu(g)$$

since ν is symmetric. Hence $\tau(\lambda \times \nu) \sim \lambda \times \nu$ iff $\lambda g \sim \lambda$ for almost every g in G [3, Theorem 1]. (Thus (M1) and (M2) imply (M).)

Conversely, if (M) holds then the set of g in G for which $a(\cdot, g)$ is a measurable automorphism of $(X, [\lambda])$ is a conull set. Because a is an action, this set is closed under multiplication and hence must be all of G [8, see Corollary 5.3]. Thus (M) implies (M1) and (M2).

Before proving that a measurable action does have an associated Borel action, we show that there can exist Borel structures compatible with the given complete measure for which the action is not Borel. In fact both changing the Borel structure and deleting a bad invariant null set may be necessary in order to get a standard Borel G-space.

EXAMPLE. Let $(X, \mathcal{B})$ be a standard Borel G-space, with a quasiinvariant probability measure λ. Let $\mathcal{M}$ be the λ-completion of $\mathcal{B}$. If $a: X \times G \to X$ is the action, then a is measurable because it is Borel. Let N be any uncountable Borel null set in X. In some cases there exist such N's meeting every orbit. For example, take an irrational winding on the 2-torus and N a circle. Since N is uncountable, there is a bijection $\theta: N \to N$ which is not a Borel function. Let

$$\mathcal{B}_\theta = \{B \subseteq X: B \setminus N \in \mathcal{B} \text{ and } \theta^{-1}(B \cap N) \in \mathcal{B}\}.$$

Then $\mathcal{B}_\theta \neq \mathcal{B}$, but $\mathcal{B} \cap \mathcal{B}_\theta$ is still dense in $\mathcal{M}$. Also, a subset of N which belongs to $\mathcal{B}_\theta$ can be translated to a subset of $X \setminus N$ which is not in $\mathcal{B}_\theta$ (look at the 2-torus again). Thus a will no longer be Borel. Since the λ-measurable sets are still the same, a is still measurable. For additional pathology, we could adjoin to X the space of cosets for some non-closed subgroup of G and give the added set measure zero. The new space is still a Lebesgue space and the action is measurable, but to get a standard Borel G-space from it the bad set must be deleted.

Our method for finding a Borel version of a measurable action involves Boolean actions [5]. Let q be the quotient mapping of $\mathcal{M}(\lambda)$ and $\mathcal{B}(X)$ onto $M(\lambda)$. Then we can define $\alpha: M(\lambda) \times G \to M(\lambda)$ by $\alpha(q(B), g) = q(a(B, g))$ for $B \in \mathcal{M}(\lambda)$. This is a valid definition provided a satisfies (M1), and α is an action of G on $M(\lambda)$, at least algebraically. Each $\alpha(\cdot, g)$ is an automorphism of $M(\lambda)$ and we have an antihomomorphism of G into $\mathrm{Aut}(M(\lambda))$. Again the question of continuity or measurability of this homomorphism arises, and first it is necessary to discuss topologies and Borel structures on $M(\lambda)$. The measure of the symmetric difference of two elements defines a metric in which $M(\lambda)$ is complete [1, page 261], but for our purposes it is more convenient to think of $M(\lambda)$ as the set of projections in $L^\infty(\lambda)$ and give $M(\lambda)$ the weak topology and Borel structure that comes from the fact that $L^\infty(\lambda)$ is the dual of $L^1(\lambda)$. This is the same as we get by thinking of $L^\infty(\lambda)$ as an algebra of multiplication operators on $L^2(\lambda)$, and taking the weak operator topology and Borel structure. About the weakest condition we could add to (M1) is *weak measurability* [5]: (M3) for each $b \in M(\lambda)$, $g \mapsto \alpha(b, g)$ is measurable from G to $M(\lambda)$. By taking linear combinations of characteristic functions and monotone limits, we see that (M3) is equivalent to:

(M3′) for b_1, b_2 in $M(\lambda)$, $g \mapsto \lambda(b_1 \wedge \alpha(b_2, g))$ is measurable.

The condition (M3) is sufficient to guarantee that a Boolean action can be realized by a point action [5, 7, 8], so we want to prove a lemma that allows us to use (M3).

LEMMA 2. *A measurable action is weakly measurable.*

PROOF. Recall the involution τ. There is a conull Borel set $A \subseteq X \times G$ such that $\tau \mid A$ is Borel. Then $A \cap \tau^{-1}(A) = (\tau \mid A)^{-1}(A)$ is also conull and Borel, so we may suppose A is invariant under τ. Then there is a Borel function $\rho: X \times G \to (0, \infty)$ which serves as $d\tau(\lambda \times \nu)/d\lambda \times \nu$. As on page 316 of [8], ρ can be modified on a null set in $X \times G$ so that $(W(g)f)(x) = \rho(x, g)^{1/2} f(xg)$ defines a measurable, and hence continuous unitary representation of G on $L^2(\lambda)$ having the canonical projection valued measure on $L^2(\lambda)$, Q as a system of imprimitivity. Then, identifying $M(\lambda)$ with the projections of $L^\infty(\lambda)$ acting on $L^2(\lambda)$, we have $\alpha(q(B), g) = W(g)^{-1} Q(B) W(g)$ for $B \in \mathcal{M}(\lambda)$ and $g \in G$. Then the joint continuity of α is clear, which at least implies (M3).

REMARK. The proof that W is measurable can also be done by the method used to prove Proposition 3.4 of [9], since a countably generated σ-algebra maps onto $M(\lambda)$. This method uses a characterization of the unitary operators $W(g)$ that does not involve the Radon–Nikodym derivatives.

Now we are ready for the positive result of this note:

THEOREM. *Let $(X, \mathcal{M}, \lambda)$ be a Lebesgue probability space, let G be a locally compact group and let $a: X \times G \to X$ be a weakly measurable action of G on X. Then there exist a conull invariant set $X_0 \subseteq X$ and a standard Borel structure $\mathcal{B}_0$ on X_0 such that*

(i) *$a \mid X_0 \times G$ is a Borel function to X_0,*

(ii) *$\mathcal{B}_0 \subseteq \mathcal{M}$ and each element of $\mathcal{M}$ differs from some element of $\mathcal{B}_0$ by a null set.*

PROOF. The plan is to take the associated Boolean action and get a point realization of it on a compact topological G-space Y. Then there will exist a point map implementing the Boolean isomorphism, and it can be changed on a null set to get a map taking a conull invariant set in Y to the desired X_0.

According to Theorem 7.5.5 of [7], since α satisfies (M3), which we extend from $M(\lambda)$ to $L^\infty(\lambda)$, there is a norm separable C^*-algebra, $\mathcal{A}$, contained in $L^\infty(\lambda)$ and containing I, such that $\alpha : \mathcal{A} \times G \to \mathcal{A}$ is norm continuous and $\mathcal{A}$ is dense in $L^\infty(\lambda)$ in the weak operator topology. Then duality gives a topological action of G on the maximal ideal space of $\mathcal{A}$, which we will denote by Y. The G-equivariant isomorphism of $C(Y)$ onto $\mathcal{A}$ extends to a system of imprimitivity for W, based on Y, which we denote by P. The range of P is $M(\lambda) \subseteq L^\infty(\lambda)$, and we denote by μ the Borel measure on Y which results: $\mu(B) = \lambda(P(B))$. Here we think of λ as being defined on $M(\lambda)$.

Next, there is a Borel function ψ_1: $Y \to X$, taking values in a conull standard subset and one-one on a conull Borel set Y_1, such that for $E \in \mathcal{M}(\lambda)$ we have $Q(E) = P(\psi_1^{-1}(E))$. This follows by an application of Satz 1 of [6] or Theorem 2.1 of [8]. Now we need to change ψ_1 slightly. First notice that for $E \in \mathcal{M}(\lambda)$ and $g \in G$, we have

$$P(\psi_1^{-1}(Eg)) = Q(Eg) = Q(E)g = P(\psi_1^{-1}(E))g = P(\psi_1^{-1}(E)g),$$

from which it follows that $\psi_1(yg) = \psi_1(y)g$ for μ-a.e. y, whenever $g \in G$, because both spaces are countably separated. Let ν be a probability measure on G equivalent to Haar measure. Since a is measurable, there is a conull Borel set $A \subseteq X \times G$ such that $a \mid A$ is Borel, and A can even be chosen to be standard. Let

$$B = \{(y, g) \in Y \times G \colon (\psi_1(y), g) \in A \text{ and } \psi_1(yg) = \psi_1(y)g\}.$$

Then B is a Borel set because $a \mid A$ is Borel and ψ_1 is Borel, and B is conull because A is conull and for each $g \in G$ we have $\psi_1(yg) = \psi_1(y)g$ for μ-a.e. y.

Let $Y_2 = \{y \in Y \colon \nu(B_y) = 1\}$ ($B_y = y$-section of B). Then Y_2 is Borel and μ-conull. Let

$$Y_0 = \{y \in Y \colon \nu(\{g \in G \colon yg \in Y_1 \cap Y_2\}) = 1\}.$$

Then it is not hard to show that Y_0 is Borel, conull and invariant. Since Y is standard, so is Y_0. If $y \in Y_2$, then for almost every g we have $\psi_1(yg)g^{-1} = \psi_1(y)$, at least for $g \in B_y$. Now let $y \in Y_0$ and choose $g_1 \in G$ so that $yg_1 \in Y_2$. Then $\psi_1(yg_1g)g^{-1} = \psi_1(yg_1)$ for almost every g, so $\psi_1(yg)g^{-1} = \psi_1(yg_1)g_1^{-1}$ for almost every g. In particular, for $y \in Y_0$ the function taking g in G to $\psi_1(yg)g^{-1}$ is essentially constant. We define $\psi(y)$ to be that constant, $\psi_1(yg_1)g_1^{-1}$. For $y \in Y_1 \cap Y_2$ we can choose $g_1 = e$ above, so $\psi(y) = \psi_1(y)$. Thus ψ is measurable, since $Y_1 \cap Y_2$ is conull, and ψ is one-one on $Y_1 \cap Y_2$. Now to show that ψ is equivariant, let $y \in Y_0$, $g_1 \in G$, and choose $g_2 \in G$ so that $yg_1g_2 \in Y_1$. Then

$$\psi(y) = \psi_1(yg_1g_2)(g_1g_2)^{-1} \quad \text{and} \quad \psi(yg_1) = \psi_1(yg_1g_2)g_2^{-1},$$

so $\psi(yg_1) = \psi(y)g_1$. Now we can show that ψ is one-one on Y_0. Let $y_1 \neq y_2$ be in Y_0. Since y_1g is almost always in $Y_1 \cap Y_2$ and the same holds for y_2g, there is at least one g such that y_1g and y_2g are both in $Y_1 \cap Y_2$. Then $\psi(y_1) = \psi_1(y_1g)g^{-1} \neq \psi_1(y_2g)g^{-1} = \psi(y_2)$.

Now ψ maps Y_0 one-one and equivariantly into X. Since $\psi_1(\mu) \sim \lambda$, the set $\psi(Y_1 \cap Y_2) = \psi_1(Y_1 \cap Y_2)$ must be conull, so $\psi(Y_0)$ is conull. Also, from $\psi_1(\mu) \sim \lambda$ it follows that each element of $\mathcal{M}(\lambda)$ is within a null set of some set $\psi(E \cap Y_1 \cap Y_2) = \psi_1(E \cap Y_1 \cap Y_2)$ for $E \in \mathcal{B}(Y)$. Thus we can let $\mathcal{B}_0 = \{\psi(E): B \in \mathcal{B}(Y_0)\}$, because $(Y_0, \mathcal{B}(Y_0))$ and $(X_0, \mathcal{B}_0)$ are isomorphic as Borel G-spaces.

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